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De Rham Cohomology of the Supermanifolds and Superstring BRST Cohomology

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Abstract

We show that the BRST operator of Neveu-Schwarz-Ramond superstring is closely related to de Rham differential on the moduli space of decorated super-Riemann surfaces $\mathcal{P}$. We develop formalism where superstring amplitudes are computed via integration of some differential forms over a section of $\mathcal{P}$ over the super moduli space $\mathcal{M}$. We show that the result of integration does not depend on the choice of section when all the states are BRST physical. Our approach is based on the geometrical theory of integration on supermanifolds of which we give a short review.

1 Introduction and summary

In this paper we show that the superstring BRST operator is closely related to the de Rham differential on the moduli space of decorated super Riemann

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surfaces. The analogous result for the bosonic string is well known [1, 2] and has numerous applications. Two important applications deserve to be mentioned. The first is the method for the calculation of string amplitudes with BRST physical states that do not have primary representatives e.g., the ghost dilaton [3, 4]. The second is the relation between the global symmetry group and BRST cohomology at ghost number one e.g., ghost number one cohomology of the critical bosonic string in the flat non-compactified background is isomorphic to the Lie algebra of Poincaré group [5]. Generalization of these applications to the superstring will be presented elsewhere [6].

The paper consists of two parts. The first is devoted to a review of the geometric theory of integration on supermanifolds. This includes differential and pseudodifferential forms, the relation between the two, their integrals and the de Rham differential. The second part contains the main result: an interpretation of the superstring measure as a differential form and a relation between the BRST operator and the de Rham differential.

2 Differential forms on supermanifolds

A common misconception in the theory of supermanifolds is that differential forms have nothing to do with the theory of integration. Indeed, a naïve generalization of differential forms to the case of a supermanifold with even coordinates x^a and odd coordinates ξ^α leads to functions $F(x, \xi; \mathrm{d}x, \mathrm{d}\xi)$ that are homogeneous polynomials in $(\mathrm{d}x, \mathrm{d}\xi)$ (note that $\mathrm{d}x$ is Grassmann odd and $\mathrm{d}\xi$ is Grassmann even) and we will see shortly that such forms cannot be integrated over supermanifolds.

In the pure even case the degree of the form can only be less or equal than the dimension of the manifold and the forms of the top degree transform as measures under smooth, orientation preserving coordinate transformations. This allows one to integrate the forms of the top degree over the oriented manifolds and forms of degree k over the oriented subspaces. On the other hand, forms on a supermanifold may have arbitrary large degree due to the presence of commuting $\mathrm{d}\xi^\alpha$ and none of them transforms as a Berezinian measure.

The correct generalization of the differential forms that can be integrated over supermanifold was first suggested by Bernstein and Leites [7]. *Pseudodifferential forms* of Bernstein and Leites, which generalize the notion of inhomogeneous differential forms (formal linear combinations of differential forms of different degree) are defined as *arbitrary generalized* functions $F(x, \xi; \mathrm{d}x, \mathrm{d}\xi)$ or distributions on $\hat{M} = \Pi TM$, where M is the manifold and Π indicates that the parity was changed to the opposite in the fibers of the tangent bundle.

We will often combine even and odd coordinates into one symbol x. Berezin integral of a pseudoform $\omega(x, \mathrm{d}x)$ over $\hat{M}$ does not depend on the choice of coordinates (two Beresinians, one from the change of x, the other from the change of $\mathrm{d}x$ cancel each other due to the parity change). We define an integral of a pseudoform ω over the manifold M as follows

$$\int_M \omega = \int_{\hat{M}} \mathcal{D}(x)\mathcal{D}(\mathrm{d}x)\, \omega(x, \mathrm{d}x), \tag{1}$$

where the integral on the right hand side is an ordinary Berezin integral. Now we can see that the integrals of pseudoforms that are polynomial in $\mathrm{d}x$ diverge unless the odd dimension of the supermanifold is zero. Pseudoforms can be integrated over arbitrary submanifolds by reducing them to the submanifold and applying eq. (1) with M replaced with the submanifold. In the case of a supermanifold with zero odd degree our integration procedure reproduces an ordinary integral of an inhomogeneous differential form which extracts the homogeneous component of the proper degree and integrates it over the submanifold.

The de Rham differential d can be defined on the pseudoforms on M simply as

$$\mathrm{d}\omega(x, \mathrm{d}x) = \sum_{A=1}^{\dim M} \mathrm{d}x^A \, \frac{\partial \omega}{\partial x^A}(x, \mathrm{d}x), \tag{2}$$

where A is a *generalized index.* For $\dim M = n|m$, index A takes n even and m odd values.

Stokes theorem is valid for the pseudoforms, yet some caution is necessary while evaluating the Berezin integral over a bounded domain (for details, see ref. [8, page 21]).

The major drawback of the theory of pseudoforms is the lack of grading, and thus the cohomology. This problem was alleviated in the geometric integration theory developed by Th. Voronov and A.V. Zorich (see ref. [8] and references therein). In this theory (homogeneous) forms on the supermanifold are presented as generalized Lagrangians of a certain type. Forms of any even and odd degree can be extracted from a pseudoform very much like homogeneous components can be extracted from inhomogeneous differential forms. This is achieved with the *Baranov-Schwartz transformation* [9, 10],

$$\lambda^{r|s} : \omega \mapsto L_\omega^{r|s}, \quad L_\omega^{r|s}(x, \dot{x}) \stackrel{\text{def}}{=} \int \mathcal{D}(\mathrm{d}t)\, \omega\Big(x, \sum_{F=1}^{r|s} \mathrm{d}t^F \dot{x}_F^A\Big). \tag{3}$$

Forms of degree $r|s$ can be integrated over non-singular subvarieties of dimension $r|s$ and a de Rham differential is defined so that it maps $r|s$-forms to $r+1|s$-forms . Both integral and differential commute with the Baranov-Schwartz transformation.

3 Superstring measure as a differential form

First, let us recall some basic facts about the operator formalism for the superstring. String states are vectors in the tensor product of a state space of a superconformal field theory and the Fock space of superconformal ghosts. The latter is generated by two supeconformal fields $B(z,\theta) = \beta(z) + \theta\, b(z)$ and $C(z,\theta) = c(z) + \theta\, \gamma(z)$.

The operator formalism associates a bra state $\langle\Sigma|$ with any punctured super Riemann surface decorated by a choice of local coordinates around each puncture. This state is to be saturated by a number of Neveu-Schwarz and Ramond string states equal to the corresponding number of punctures on Σ.

Given a local superconformal vector field $V^{(i)}(z,\theta)$ for each puncture on Σ, one can define a Schiffer variation $\delta_V \Sigma$. Under a Schiffer variation the bra state $\langle\Sigma|$ transforms as follows

$$\delta_V \langle\Sigma| = \langle\Sigma|\langle \mathbf{T}, V\rangle, \tag{4}$$

where $\mathbf{T}(z,\theta) = \frac{1}{2}G(z) + \theta\, T(z)$ is the total superconformal energy-momentum tensor and the pairing $\langle \mathbf{T}, V\rangle$ is given by

$$\langle \mathbf{T}, V\rangle = \sum_{\text{punctures}} \oint \mathrm{d}z\mathrm{d}\theta\, \mathbf{T}^{(i)}(z,\theta) V^{(i)}(z,\theta). \tag{5}$$

Another important property of $\langle\Sigma|$ is that $\langle\Sigma|Q = 0$ where $Q = \sum Q^{(i)}$ is the sum of BRST operators, one for each puncture.

Let $|\psi_1\rangle, |\psi_2\rangle \cdots |\psi_n\rangle$ be string states that saturate $\langle\Sigma|$ and $|\Psi\rangle$ be their tensor product. The string amplitude is given by an integral over a set of

Σ's that covers the moduli space $\mathcal{M}$ of punctured super Riemann surfaces. Let σ be a section of $\mathcal{P}$ over $\mathcal{M}$, then

$$\langle\!\langle \psi_1\, \psi_2 \cdots \psi_n \rangle\!\rangle = \int_{\sigma(\mathcal{M})} \mu_\Psi. \tag{6}$$

Where μ_Ψ is the *superstring measure* given by the following expression [11]

$$\mu_\Psi(V_F;\Sigma) = \langle\Sigma|\delta^{d_e|d_o}(\,\langle B, V_F\rangle\,)|\Psi\rangle, \tag{7}$$

where $d_e|d_o$ is the dimension of the super moduli space and $F = 1, \ldots, d_e|d_o$. For genus g surfaces with p Neveu-Schwarz and q Ramond punctures $d_e|d_o = 3g-3+p+q|2g-2+p+\frac{q}{2}$. The pairing $\langle B, V_F\rangle$ is defined as in eq. (5) by

$$\langle B, V\rangle = \sum_{\text{punctures}} \oint \mathrm{d}z\mathrm{d}\theta\, B^{(i)}(z,\theta)V^{(i)}(z,\theta); \tag{8}$$

moreover, the two are related by $\{Q, \langle B, V_F\rangle\} = \langle \mathbf{T}, V_F\rangle$.

Substituting the delta function by its integral representation we immediately recognize a $d_e|d_o$-form which is a Baranov-Schwartz transform of

$$\omega_\Psi(\Sigma, \mathrm{d}\Sigma) = \langle\Sigma|e^{i\langle B, \mathrm{d}\Sigma\rangle}|\Psi\rangle; \tag{9}$$

in other words,

$$\mu_\Psi = \lambda^{d_e|d_o}\,\omega_\Psi. \tag{10}$$

The main property of ω_Ψ which follows easily from the definitions is that

$$\mathrm{d}\,\omega_\Psi = \omega_{Q\Psi}, \tag{11}$$

This property implies immediately that the string amplitude (6) does not depend on σ or on the choice of local coordinates when the scattering states are BRST physical, and also that it does not change when the states are changed by adding BRST trivial vectors:

$$\int_{\sigma(\mathcal{M})} \mu_\Psi = \int_{\sigma'(\mathcal{M})} \mu_\Psi, \tag{12}$$

and

$$\int_{\sigma(\mathcal{M})} \mu_\Psi = \int_{\sigma(\mathcal{M})} \mu_{\Psi+Q\Lambda}. \tag{13}$$

Both eqs. (12) and (13) are direct consequences of the main identity (11) and the Stokes theorem. Another way to interpret eq. (11) is to say that eqs. (9) and (10) define a natural map from BRST cohomology to de Rham cohomology of $\mathcal{P}$.

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